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Mapping Class Groups of 3-Manifolds, Then and Now

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Abstract. Six of a set of seven conjectures about 3-manifold mapping class groups proposed in the 1990's by the second author are proven for orientable 3-manifolds using the Geometrization Theorem and other subsequent results about 3-manifolds. We prove the seventh conjecture for some cases, and also develop some results that refine one of the proven ones.

Introduction

The mapping class group of a manifold M is the group $\mathcal{H}(M)$ of isotopy classes of homeomorphisms (we do not require that the homeomorphisms be orientation-preserving). In the 1990's, the second author proposed a set of conjectures concerning mapping class groups of 3-manifolds, that were included as Problem 3.49 of the Kirby problem list [**17**]. In the intervening years, some remarkable advances have been made in low-dimensional topology, culminating in the proof of Thurston's Geometrization Conjecture by Perelman. We will see that at least in the orientable case most of the conjectures follow from these newer results. The nonorientable versions remain unresolved, although it seems likely that the conjectures hold for them as well.

Problem 3.49 will be reproduced in its entirety in Section 1, where we will see that it consists of seven conjectures called A through G. Section 2 gives additional comments and examples delineating them. In Section 4, we will see that the first six of the conjectures have now been established (Conjecture B requires an additional hypothesis to eliminate simple counterexamples when the manifold has 2-sphere boundary components). Conjecture G is a longer story, given in Section 5: it is established in many cases, but remains open in general. Finally, in Section 6, we prove some results that extend Conjecture B.

The remaining section of the paper, Section 3, proves the Finite Mapping Class Group Theorem, basically the result that the mapping class groups of closed orientable irreducible non-Haken 3-manifolds are finite. It is a key ingredient in the proofs of several of the conjectures and subsequent results. As we will see, it follows rather easily using the Geometrization Theorem and a major result of D. Gabai, R. Meyerhoff, and N. Thurston [**10**] from the early 2000's, together with earlier work of P. Scott, M. Boileau, and J.-P. Otal.

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The authors are grateful to the referee for suggesting the inclusion of additional examples illustrating some of the conjectures. This prompted us to add Section 2 to the revision of the manuscript.

In the remainder of this article, all 3-manifolds will be assumed to be orientable.

1. The conjectures

Here are the conjectures as given in [**17**]:

Problem 3.49 (McCullough) Generalizing the construction of Dehn twist homeomorphisms of 2-manifolds, define a *Dehn homeomorphism* as follows: Let $(F^{n-1} \times I, \partial F^{n-1} \times I) \subset (M^n, \partial M^n)$, where F is a connected codimension-1 submanifold and $F \times I \cap \partial M = \partial F \times I$. Let φ_t be an element of $\pi_1(\mathrm{Homeo}(F), 1_F)$, i.e. for $0 \leq t \leq 1$, φ_t is a continuous family of homeomorphisms of F such that $\varphi_0 = \varphi_1 = 1_F$. Define $h \in \pi_0(\mathrm{Homeo}(M)) = \mathcal{H}(M)$ by

$$h(x,t) = \begin{cases} (\varphi_t(x), t) & \text{if } (x,t) \in F \times I, \\ h(m) = m & \text{if } m \notin F \times I. \end{cases}$$

Note that when $\pi_1(\mathrm{Homeo}(F))$ is trivial, a Dehn homeomorphism must be isotopic to the identity. Define the *Dehn subgroup* $\mathcal{D}(M)$ of $\mathcal{H}(M)$ to be the subgroup generated by Dehn homeomorphisms.

The following table lists $\pi_1(\mathrm{Homeo}(F))$ for connected 2-manifolds, and the names of the corresponding Dehn homeomorphisms of 3-manifolds.

F	$\pi_1(\mathrm{Homeo}(F))$	Dehn homeomorphism
$S^1 \times S^1$	$\mathbb{Z} \times \mathbb{Z}$	*Dehn twist about a torus*
$S^1 \times I$	$\mathbb{Z}$	*Dehn twist about an annulus*
D^2	$\mathbb{Z}$	*twist*
S^2	$\mathbb{Z}/2\mathbb{Z}$	*rotation about a sphere*
$\mathbb{RP}^2$	$\mathbb{Z}/2\mathbb{Z}$	*rotation about a projective plane*
Klein bottle	$\mathbb{Z}$	*Dehn twist about a Klein bottle*
Möbius band	$\mathbb{Z}$	*Dehn twist about a Möbius band*
$\chi(F) < 0$	$\{0\}$	

(A) **Dehn Subgroup Conjecture:** *Let M be a compact 3-manifold. Then $\mathcal{D}(M)$ has finite index in $\mathcal{H}(M)$.*

Remarks: For M orientable, (A) is true if it is true for irreducible manifolds [**23**]. Johannson [**20**, Corollary 27.6] proved (A) for boundary-irreducible Haken manifolds, and this was extended to all Haken manifolds in [**27**].

Denote by $\mathcal{D}_{>0}(M)$ the subgroup of $\mathcal{D}(M)$ generated by Dehn homeomorphisms using D^2, S^2, and $\mathbb{RP}^2$ (the surfaces of positive Euler characteristic).

By an argument similar to the proof of Proposition 1.2 of [**24**], one can prove that if ∂M is incompressible, then $\mathcal{D}_{>0}(M)$ is a finite abelian group.

When the boundary of M is compressible, the following results were proved in [**22**]:

- If ∂M is almost incompressible, then $\mathcal{D}_{>0}(M)$ is a finitely generated abelian group (almost incompressible means that in each boundary component F of M, there is at most one simple closed curve up to isotopy that bounds a disk in M but does not bound a disk or Möbius band in F);

- If ∂M is not almost incompressible, then $\mathcal{D}_{>0}(M)$ is infinitely generated and nonabelian.

(B) **Kernel Conjecture:** *$\mathcal{D}_{>0}(M)$ has finite index in the kernel of $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$.*

Remarks: In general, $\mathcal{D}_{>0}(M)$ need not equal the kernel, as shown by the example of reflection in the fibers of an I-bundle. For orientable M containing no fake 3-cells, (B) is true if it is true for irreducible M [**24**]. The main case in which (B) is unknown is when M is irreducible, aspherical and not sufficiently large, although even here some cases are known by work of D. Gabai [**8**, **9**].

Define $\mathrm{Out}_{\partial M}(\pi_1(M))$ to be the subgroup of $\mathrm{Out}(\pi_1(M))$ consisting of the automorphisms φ such that for every boundary component F of M, there exists a boundary component G so that $\varphi(i_\#(\pi_1(F)))$ is conjugate in $\pi_1(M)$ to $j_\#(\pi_1(G))$, where $i\colon F \to M$ and $j\colon G \to M$ are the inclusions. This subgroup contains the image of $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$.

(C) **Image Conjecture:** *The homomorphism $\mathcal{H}(M) \to \mathrm{Out}_{\partial M}(\pi_1(M))$ has image of finite index.*
Remarks: In general, the image is not all of $\mathrm{Out}_{\partial M}(\pi_1(M))$ (see the discussion in the next section). Again, (C) is true if it is true for irreducible manifolds [**24**].

(B) and (C) combine to give the following conjecture, where *almost exact* means that images have finite indexes in kernels (rather than equaling kernels as in exactness).

(D) **Almost Exactness Conjecture:** *Let M be a compact 3-manifold. Then the sequence*

$$1 \to \mathcal{D}_{>0}(M) \to \mathcal{H}(M) \to \mathrm{Out}_{\partial M}(\pi_1(M)) \to 1$$

is almost exact.

(E) **Finiteness Conjecture:** *Let M be closed, irreducible, but not sufficiently large. Then $\mathcal{H}(M)$ is finite.*
Remarks: Note that (E) follows from the Dehn Subgroup Conjecture (A). (E) has been proven by Gabai for many aspherical but not sufficiently large manifolds [**8**, **9**]. Also, $\mathcal{H}(M)$ should be finite when $M = S^3/G$ for $G \in \mathrm{SO}(4)$ for then it is conjectured that $\mathcal{H}(M) = \pi_0(\mathrm{Isom}(M))$ (Problem 3.47).

(F) **Finite Presentation Conjecture:** *$\mathcal{H}(M)$ is finitely presented.*
Remarks: For orientable M, (F) is true if it is true for irreducible manifolds [**16**], and is known in many cases, for example lens spaces [**4**] and Haken manifolds [**11**, **33**].

Recall that a group is said to have a property virtually if some finite-index subgroup has the property.

(G) **Virtual Geometric Finiteness Conjecture:** *Let M be a compact 3-manifold. Then*
 (i) $\mathcal{H}(M)$ *is virtually torsion-free.*
 (ii) $\mathcal{H}(M)$ *is virtually of finite cohomological dimension.*
 (iii) $\mathcal{H}(M)$ *is virtually geometrically finite (a group is geometrically finite if it is the fundamental group of a finite aspherical complex).*

Remarks: Since (iii) implies (ii) and (ii) implies (i), this is really a sequence of three successively stronger conjectures. All hold for compact 2-manifolds by work of J. Harer [**12**, **13**] and W. Harvey [**14**, **15**], and for Haken manifolds [**25**], and hold trivially in the cases where the mapping class group is known to be finite. For non-irreducible 3-manifolds, the following is a preliminary question. Define the rotation subgroup $\mathcal{R}(M)$ to be the subgroup generated by rotations about 2-spheres and 2-sided projective planes in M; it is a finite normal abelian subgroup of $\mathcal{H}(M)$. Is there a finite-index subgroup of $\mathcal{H}(M)$ that intersects $\mathcal{R}(M)$ trivially? If not, replace $\mathcal{H}(M)$ by $\mathcal{H}(M)/\mathcal{R}(M)$ in the conjecture.

2. Additional remarks and examples for the Conjectures

The Dehn subgroup $\mathcal{D}(M)$ is contained in the subgroup $\mathcal{H}_+(M)$ consisting of the orientation-preserving elements of $\mathcal{H}(M)$. Since Dehn homeomorphisms preserve each boundary component, $\mathcal{D}(M)$ lies in the subgroup of $\mathcal{H}_+(M)$ that acts trivially on the set of boundary components, but even in this subgroup, it may have large index. For example, $\mathcal{D}(M)$ is trivial for any compact 3-manifold whose interior admits a complete hyperbolic metric of finite volume, yet by a result of S. Kojima [**18**], every finite group occurs as the full isometry group (and hence as $\mathcal{H}(M)$, by results detailed in the next section) for some closed hyperbolic 3-manifold.

The image of $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is typically a subgroup of infinite index. Examples abound when M has a nonseparating compressing disk and is not a compression body. For then, one may take a 1-handle C and map it around a nonperipheral loop ℓ in $\overline{M - C}$, then over C. The homotopy inverse takes C around the reverse of ℓ and over C in the same direction. Apart from a few exceptional "small" cases, fully analyzed in Main Topological Theorem 1 of [**6**], the automorphisms induced by the powers of such a homotopy equivalence and all its powers represent distinct cosets of the image of $\mathcal{H}(M)$ in $\mathrm{Out}(\pi_1(M))$. Main Topological Theorem 2 of the same work analyzes the rather complicated case of incompressible boundary; roughly speaking, the index is infinite unless all components of the characteristic submanifold that meet the boundary are of certain "small" types.

Conjecture (C) can be interpreted as saying that all the phenomena that allow the image of $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ to have infinite index involve the boundary. Even in the closed case, however, the image need not be all of $\mathrm{Out}_{\partial M}(\pi_1(M))$. One type of example is any connected sum $L(m, q_1) \# L(m, q_2)$ for which the summands are not homeomorphic. The fundamental group is the free product $\mathbb{Z}/m * \mathbb{Z}/m$, but no outer automorphism that interchanges the summands can be induced by a homeomorphism. In the irreducible case, a lens space $L(m, q)$ may admit many outer automorphisms of the fundamental group— multiplications by any nontrivial element which has a multiplicative inverse modulo m— but as proven by F. Bonahon [**4**], its mapping class group never has order larger than 4. More subtle examples with finite fundamental group were obtained by S. Plotnick [**29**]. For example if Σ is the Poincaré sphere, then the unique nontrivial outer automorphism of $\mathrm{Out}(\pi_1(\Sigma))$ is not induced by any homeomorphism. On the other hand, $\mathcal{H}(M) \to \mathrm{Out}_{\partial M}(\pi_1(M))$ is surjective for aspherical 3-manifolds. This holds by Waldhausen's celebrated work [**32**] in the Haken case, by Mostow Rigidity in the hyperbolic case, and follows from results in the literature for the non-Haken non-hyperbolic cases (see [**28**, Proposition 7.1]).

3. The Finite Mapping Class Group Theorem

Most of the Conjectures are resolved, at least for the orientable case, by Perelman's completion of the proof of the Geometrization Theorem and other fundamental work. The main step is the following:

THEOREM (Finite Mapping Class Group Theorem). *Let M be a closed orientable irreducible non-Haken 3-manifold. Then $\mathcal{H}(M)$ and $\mathrm{Out}(\pi_1(M))$ are finite, and if M is not S^3 or $\mathbb{RP}^3$, then $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is injective.*

PROOF. When $\pi_1(M)$ and hence $\mathrm{Out}(\pi_1(M))$ are finite, the Geometrization Theorem implies that M is the quotient of S^3 by a finite group of isometries. For these manifolds, as detailed in the proof of Theorem 3.1 in [**26**], the work of many authors shows that apart from S^3 and $\mathbb{RP}^3$, $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is injective.

When $\pi_1(M)$ is infinite, we appeal to the Geometrization Theorem again to deduce that every non-Haken irreducible orientable 3-manifold with infinite fundamental group is either a Seifert-fibered space or a hyperbolic manifold.

Gabai, Meyerhoff, and N. Thurston [**10**] proved that $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is an isomorphism for closed hyperbolic 3-manifolds. For any closed hyperbolic n-manifold with $n \geq 3$, $\mathrm{Out}(\pi_1(M))$ is finite by Mostow Rigidity (see R. Benedetti and C. Petronio [**1**, Theorem C.5.6]).

The non-Haken Seifert manifolds with infinite fundamental group fiber over S^2 with exactly three exceptional fibers, and for all such manifolds, $\mathrm{Out}(\pi_1(M))$ is finite [**25**, p. 21]. Scott [**30**] and Boileau and Otal [**2**, **3**] showed that $\mathcal{H}(M) \to \mathrm{Out}(\pi_1(M))$ is injective in all such cases (see also T. Soma [**31**, Theorem 0.2]). □

4. Conjectures A through F

Recall that we are assuming throughout that M is orientable.

As explained in the remarks to Conjecture A, the conjecture is known to follow from the non-Haken irreducible case, which is immediate from the Finite Mapping Class Group Theorem.

In Conjecture B, the assumption that M has no 2-sphere boundary components should be added to the statement, since otherwise "slide homeomorphisms" (defined below) that move a D^3-summand around an arc in M can occur. In fact, with this assumption, $\mathcal{D}_{>0}(M)$ is the full kernel of $\mathcal{H}_+(M) \to \mathrm{Out}(\pi_1(M))$, where as before, $\mathcal{H}_+(M)$ is the orientation-preserving subgroup of $\mathcal{H}(M)$:

THEOREM 4.1. *If M has no 2-sphere boundary components, then $\mathcal{D}_{>0}(M)$ equals the kernel of $\mathcal{H}_+(M) \to \mathrm{Out}(\pi_1(M))$.*

PROOF. We first recall the definition of slide homeomorphisms. Suppose that M is a connected sum $M_1 \# M_2 = \overline{M_1 - D^3} \cup_\Sigma \overline{M_2 - D^3}$, where Σ is a 2-sphere. We write M_i' for $\overline{M_i - D^3}$. Let α be an arc in M_2' meeting Σ exactly in its endpoints.

A *slide of* M_1 *around* α is defined as follows. Let N be the manifold from M_2' obtained by filling the boundary component Σ of M_2' with a 3-ball E. Let W be a regular neighborhood of $E \cup \alpha$, a solid torus in N. Choose an isotopy J_t of N such that:

(1) $J_0 = 1_N$ and $J_1|_E = 1_E$.
(2) $J_t(x) = x$ for $0 \leq t \leq 1$ and $x \notin W$.
(3) During the isotopy J_t, E travels once around the loop α.

Now define $h\colon M \to M$ by

$$h(x) = \begin{cases} x & x \in M_1' \\ J_1(x) & x \in M_2' \end{cases}$$

We remark that h is isotopic to a Dehn twist about the torus ∂W in M, provided that one chooses J so as not to introduce an additional rotation about Σ.

To prove the theorem, we recall that Theorem 1.5 of [**24**] gives the following set of generators for the kernel of $\mathcal{H}_+(M) \to \mathrm{Out}(\pi_1(M))$:

(1) interchanges of D^3-summands,
(2) slides of D^3-summands,
(3) interchanges of fake 3-cell summands,
(4) slides of fake 3-cell summands, and
(5) Dehn twists about 2-spheres,
(6) homeomorphisms supported on one irreducible summand N which induce the identity automorphism on $\pi_1(N)$.

The assumption that M has no 2-sphere boundary components eliminates generators of types (1) and (2), and the Geometrization Theorem eliminates types (3) and (4). Type (5) are in $\mathcal{D}_{>0}(M)$, so we consider those of type (6) on an irreducible orientable summand N. It is sufficient to consider only orientation-preserving elements of the kernel (orientation-reversing elements can exist, specifically reflection in the fibers of I-bundles, and reflections of S^3 and $\mathbb{RP}^3$).

If N is Haken, then Theorem 6.2.1 of [**27**] shows that Dehn twists about disks generate the kernel, and these lie in $\mathcal{D}_{>0}(M)$.

Suppose that N is non-Haken. The Finite Mapping Class Group Theorem shows that there are no nontrivial elements of type (6), unless N is S^3 or $\mathbb{RP}^3$. But in these two cases, the nontrivial elements inducing the identity automorphism are orientation-reversing, so do not produce elements of $\mathcal{H}_+(M)$. □

For Conjectures C and F, as in the original Remarks, it is sufficient to consider irreducible M, in which case they hold by Waldhausen's results [**32**] in the Haken case and by the Finite Mapping Class Group Theorem in the non-Haken case.

Conjecture D is immediate from Conjectures B and C, and Conjecture E is part of the Finite Mapping Class Group Theorem.

5. Conjecture G

As noted in the remarks, Conjecture G holds for Haken 3-manifolds. The Finite Mapping Class Group Theorem shows that it holds for the non-Haken and hence for all irreducible cases.

The question of whether $\mathcal{H}(M)$ has a finite-index subgroup that meets $\mathcal{R}(M)$ trivially is still open, but we can now prove the weaker form of Conjecture G(ii) in the closed case:

THEOREM 5.1. *Let M be a closed orientable 3-manifold. Then $\mathcal{H}(M)/\mathcal{R}(M)$ has finite virtual cohomological dimension.*

PROOF. It suffices to prove the theorem for $\mathcal{H}_+(M)/\mathcal{R}(M)$. Theorem 4.1 shows that $\mathcal{H}_+(M)/\mathcal{R}(M) \to \mathrm{Out}(\pi_1(M))$ is injective. By Conjecture C, it has image of finite index. Consequently, to prove the theorem, it suffices to prove that $\mathrm{Out}(\pi_1(M))$ has finite virtual cohomological dimension. To achieve this, we will apply the following theorem of V. Guirardel and G. Levitt [**19**, Corollary 5.3]:

THEOREM. *Let G be a free product $G_1 * \cdots * G_p * F_k$, with each G_i indecomposable and with F_k free.*

(1) *If each G_i has a subgroup H_i of finite index with H_i and $H_i/Z(H_i)$ torsion-free, and $\mathrm{Out}(H_i)$ virtually torsion-free, then $\mathrm{Out}(G)$ is virtually torsion-free.*
(2) *If furthermore H_i and $\mathrm{Out}(H_i)$ have finite virtual cohomological dimension, then $\mathrm{Out}(G)$ has finite virtual cohomological dimension.*

In our case, $G = \pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n)$, where each M_i is a prime summand of M, and we take each $G_i = \pi_1(M_i)$. To complete the proof, we must verify the hypotheses of the Guirardel-Levitt theorem.

If $\pi_1(M_i)$ is finite, then we may take H_i trivial, and if it is infinite cyclic, we take $H_i = \pi_1(M_i)$. So we may assume that M_i is aspherical.

Suppose first that M_i is Seifert-fibered. If M_i is the 3-torus, then we may take $H_i = \pi_1(M_i) = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, with $\mathrm{Out}(\pi_1(M_i)) = \mathrm{GL}(3,\mathbb{Z})$ of finite virtual cohomological dimension by a theorem of A. Borel and J.-P. Serre [**5**]. Otherwise, M_i admits a finite covering by a circle bundle $\widetilde{M_i}$ over a closed orientable 2-manifold F_i of genus at least 1. Take $H_i = \pi_1(\widetilde{M_i})$. Since $\widetilde{M_i}$ is a circle bundle but not the 3-torus, there is a central extension

$$1 \to \mathbb{Z} \to \pi_1(\widetilde{M_i}) \to \pi_1(F_i) \to 1 ,$$

so $H_i/Z(H_i) = \pi_1(F_i)$ is torsion-free. Since $\widetilde{M_i}$ is Haken, we have $\mathrm{Out}(H_i) \cong \mathcal{H}(\widetilde{M_i})$ by F. Waldhausen's result [**32**], and $\mathcal{H}(\widetilde{M_i})$ has finite virtual cohomological dimension [**25**].

The remaining case is when M_i is aspherical and not Seifert-fibered. We take $H_i = \pi_1(M_i)$, which is torsion-free, centerless, and has finite virtual cohomological dimension. If M_i is Haken, then again $\mathrm{Out}(\pi_1(M_i))$ has finite virtual cohomological dimension by [**25**], and if M_i is non-Haken, then $\mathrm{Out}(\pi_1(M_i))$ is finite by the Finite Mapping Class Group Theorem. □

Now we consider the case when M may have nonempty boundary. Under some hypotheses, we can prove the weakest form of Conjecture G(iii):

THEOREM 5.2. *Let M be a compact orientable 3-manifold with no 2-sphere boundary components and incompressible boundary. Assume that each irreducible summand M_i of M has the property that $\mathcal{H}(M_i) \to Out(M_i)$ has image of finite index. Then $\mathcal{H}(M)/\mathcal{R}(M)$ is virtually torsion-free.*

It is known exactly which irreducible M_i have the property that $\mathcal{H}(M_i) \to \mathrm{Out}(M_i)$ has image of finite index. By Waldhausen's theorem [**32**] in the Haken case and the Finite Mapping Class Group Theorem in the non-Haken case, all closed irreducible M_i satisfy this. When M_i has nonempty incompressible boundary, the condition is that the Seifert-fibered components of the characteristic submanifold that meet ∂M_i must be rather small; exact conditions are given in Main Topological Theorem 1 (absolute case) in Canary and McCullough [**6**]. When M_i has compressible boundary, M_i must be a compression body or else be one of some very simple types given in the Main Topological Theorem 2 (absolute case) of Canary and McCullough [**6**].

Proof of Theorem 5.2. It suffices to prove that $\mathcal{H}_+(M)/\mathcal{R}(M)$ is virtually torsion-free. Since M has no essential compressing disks, Theorem 4.1 shows that $\mathcal{R}(M)$ is the kernel of $\mathcal{H}_+(M) \to \mathrm{Out}(\pi_1(M))$, so it suffices to show that $\mathrm{Out}(\pi_1(M))$ is virtually torsion-free. As in Theorem 5.1, we will apply the theorem of Guirardel and Levitt, with $G_i = \pi_1(M_i)$, although we will only need to verify the torsion-free hypotheses. As in the proof of that theorem, the hypotheses are satisfied when M_i is closed, so it remains to verify them when M_i has nonempty boundary.

If M_i is Seifert-fibered, then it has a finite covering $\widetilde{M_i} \to M_i$, where now $\widetilde{M_i}$ is a product $F_i \times S^1$ with F_i an aspherical orientable surface, and we take $H_i = \pi_1(\widetilde{M_i})$, which is torsion-free.

Suppose first that F_i is an annulus. Then $\pi_1(\widetilde{M_i}) = \mathbb{Z}\times\mathbb{Z}$ so $H_i/Z(H_i)$ is trivial and $\mathrm{Out}(H_i) \cong \mathrm{GL}(2,\mathbb{Z})$ is virtually free.

If F_i is not an annulus, then $H_i/Z(H_i) = \pi_1(F_i)$ is free of rank at least 2, hence torsion-free. For $\mathrm{Out}(H_i)$, we regard H_i as a direct product $F \times \mathbb{Z}$ where F is free of rank at least 2. The $\mathbb{Z}$-factor is characteristic, and fixed by an index-2 subgroup of $\mathrm{Out}(F \times \mathbb{Z})$. There is a surjection from this subgroup to $\mathrm{Out}(F)$, and $\mathrm{Out}(F)$ is virtually torsion-free by work of M. Culler and K. Vogtmann [**7**]. Since the kernel $\mathrm{Hom}(F,\mathbb{Z}) \cong H^1(F)$ of this surjection is torsion-free, $\mathrm{Out}(F \times \mathbb{Z})$ is also virtually torsion-free. □

6. Beyond Conjecture B

In this section, as throughout, we assume that M is orientable. In this case, $\mathcal{D}_{>0}(M)$ is the subgroup of $\mathcal{H}(M)$ generated by twists about disks and rotations about 2-sphere. For orientable M, we can analyze the group $\mathcal{D}_{>0}(M)$ in more depth; in particular, we will show that $\mathcal{D}_{>0}(M)/\mathcal{R}(M)$ is torsion-free. This gives Theorem 6.3 below, a companion result to Theorem 4.1.

First we will need a result about the *twist group* $\mathcal{T}(M)$, which is the subgroup of $\mathcal{H}(M)$ generated by twist homeomorphisms. The statement involves the characteristic compression body. In an irreducible orientable 3-manifold each compressible boundary component has a closed neighborhood which is a compression body (with incompressible frontier), and the neighborhood is unique up to isotopy. Details are given in [**27**], and for a more general context in [**6**, Chapter 3]. In a reducible 3-manifold, the compression body neighborhood may no longer be unique up to isotopy, but it does have incompressible frontier (possibly having 2-sphere components), which is the only property needed in the next theorem.

THEOREM 6.1. *Let M be orientable with nonempty compressible boundary, and let $V_1, \ldots\ V_k$ be disjoint characteristic compression body neighborhoods of the compressible boundary components $F_1, \ldots,\ F_k$ of M. Then $\mathcal{D}_{>0}(M) = \mathcal{T}(M) = \mathcal{R}(M) \times (\prod_{1\leq i\leq k} \mathcal{T}(V_i))$.*

PROOF. Fix a connected-sum decomposition $M = M_1\#\cdots\#M_n$ along a collection of disjoint imbedded 2-spheres S_k such that each M_j is prime. Enlarge the collection of S_k to include an S^2-fiber from each $S^2 \times S^1$ summand of M. We may assume that each V_i lies in some M_j. As in Proposition 3.1.3(b) of [**22**], $\mathcal{R}(M)$ is generated by rotation homeomorphisms about the S_k.

From Proposition 3.1.3 of [**22**], $\mathcal{R}(M)$ is a central normal subgroup of $\mathcal{H}(M)$. Moreover it is contained in $\mathcal{T}(M)$, and consequently $\mathcal{D}_{>0}(M) = \mathcal{T}(M)$.

Let D be any (essential) compressing disk in M, with $\partial D \subset \partial V$ for some $V \in \{V_i\}$. Write t_D for a Dehn twist about D. Let G be the frontier of V. We may assume that D is transverse to G, and let $E \subset G$ be a disk bounding an innermost intersection circle. Surgery along E produces a disk D' and a 2-sphere S disjoint from D', and t_D is isotopic to the composition of $t_{D'}$ and a rotation about S. Since D' has fewer intersections with G than did D, we may repeat this process until we produce a disk $D_V \subset V$ such that t_D is isotopic to the composition of t_{D_V} with an element of $\mathcal{R}(M)$.

There is a natural homomorphism from $\mathcal{T}(V_i)$ to $\mathcal{T}(M)$. Using these together with the inclusion of $\mathcal{R}(M)$ into $\mathcal{H}(M)$ defines a homomorphism

$$\Phi\colon\ \mathcal{R}(M) \times (\prod_{1\leq i\leq k} \mathcal{T}(V_i)) \to \mathcal{T}(M)\ .$$

The previous paragraph shows that Φ is surjective. Suppose that ρ is an element of the kernel. Then the restriction of $\Phi(\rho)$ to ∂M is isotopic to the identity. By Proposition 2.11(b) of [**22**], the restriction of $\mathcal{T}(V_i)$ to $\mathcal{H}(F_i)$ is injective, so the $\mathcal{T}(V_i)$-coordinates of ρ are trivial. Therefore ρ lies in $\mathcal{R}(M)$, but the inclusion of $\mathcal{R}(M) \to \mathcal{T}(M) \to \mathcal{H}(M)$ is injective by definition, so ρ is trivial. Therefore Φ is an isomorphism. □

THEOREM 6.2. *If M is orientable, then $\mathcal{D}_{>0}(M)/\,\mathcal{R}(M)$ is torsion-free.*

PROOF. If ∂M is incompressible, then $\mathcal{D}_{>0}(M) = \mathcal{R}(M)$, so assume that ∂M is compressible, and choose $V_1, \ldots,\ V_k$ as in Theorem 6.1. By that result, $\mathcal{D}_{>0}(M)/\,\mathcal{R}(M) \cong \prod_{1\leq i\leq k} \mathcal{T}(V_i)$, so it suffices to show that each $\mathcal{T}(V_i)$ is torsion-free.

If $V_i \cap \partial M$ is a torus, then V_i is a solid torus and $\mathcal{T}(V_i)$ is infinite cyclic. Assume, then, that $V_i \cap \partial M$ has genus at least 2. Elements of $\mathcal{T}(V_i)$ lie in the subgroup of elements of $\mathcal{H}(V_i)$ that are homotopic to the identity (in fact, $\mathcal{T}(V_i)$ equals this subgroup, but we do not need that fact). The results B. Maskit in [**21**] (in particular, Corollary 7), when translated into our 3-manifold language, say that this subgroup acts freely on a Teichmüller space, and consequently it must be torsion-free. □

THEOREM 6.3. *Let M be compact and orientable, with no 2-sphere boundary components. Then the kernel of*

$$\Phi\colon\ \mathcal{H}_+(M)/\,\mathcal{R}(M) \to \mathrm{Out}(\pi_1(M))$$

is torsion-free.

PROOF. By Theorem 4.1, $\mathcal{D}_{>0}(M)$ is the kernel of $\mathcal{H}_+(M) \to \mathrm{Out}(\pi_1(M))$, so $\mathcal{D}_{>0}(M)/\mathcal{R}(M)$ is the kernel of $\mathcal{H}_+(M)/\mathcal{R}(M) \to \mathrm{Out}(\pi_1(M))$. If ∂M is incompressible, then $\mathcal{D}_{>0}(M)/\mathcal{R}(M)$ is trivial, and if ∂M is compressible then Theorem 6.2 shows that it is torsion-free. □

Finally, we remark that in almost all cases, the kernel of $\mathcal{H}(M)/\mathcal{R}(M) \to \mathrm{Out}(\pi_1(M))$ equals the kernel of $\mathcal{H}_+(M)/\mathcal{R}(M) \to \mathrm{Out}(\pi_1(M))$. Theorem 4.3.4, adapted to the orientable case and the post-Perelman age, gives the exceptions:

THEOREM 6.4. *Let M be a compact orientable 3-manifold. Then M admits an orientation-reversing homeomorphism that induces the identity outer automorphism on $\pi_1(M)$ if and only if either M is S^3 or every prime summand of M is one of $S^2 \times S^1$, an I-bundle, or $\mathbb{RP}^3$.*

For the exceptional manifolds described in Theorem 6.4, $\mathcal{H}(M)$ does have an orientation-reversing element of order 2, so the kernel of $\mathcal{H}(M)/\mathcal{R}(M) \to \mathrm{Out}(\pi_1(M))$ is not torsion-free.

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